Indefinite Integrals/Applications of The Fundamental Theorem

We saw last time that if we can find an antiderivative for a continuous function f, then we can evaluate the integral

$$\int_a^b f(x)dx.$$

Indefinite Integrals

In light of the relationship between the antiderivative and the integral above, we will introduce the following (traditional) notation for antiderivatives:

$$\int f(x)dx = F(x) + C, \quad \text{means that} \quad F'(x) = f(x) \text{ and } C \text{ is a constant}$$

This family of functions $\int f(x)dx$, is called **the indefinite integral** (of f). We refer to the integral $\int_a^b f(x)dx$ as **the definite integral**. Note that the definite integral is a number whereas the indefinite integral refers to a family of functions.

We begin by making a list of the antiderivatives we know and the elementary rules governing the calculation of antiderivatives, which we get by reversing our previous lists of derivatives and rules:

$\int cf(x)dx = c\int f(x)dx$	$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int kdx = kx + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \sin xdx = -\cos x + C$	$\int \cos xdx = \sin x + C$
$\int \sec^2 xdx = \tan x + C$	$\int \csc^2 xdx = -\cot x + C$
$\int \sec x \tan xdx = \sec x + C$	$\int \csc x \cot xdx = -\csc x + C$

Note that $1/x^n, n > 1$ and $\tan x$ are not continuous functions. In the case of non-continuous functions it is understood that the antiderivatives differ by a constant on each interval where the function is continuous. For example for the function $1/x^2$, we have in reality

$$\int \frac{1}{x^2}dx = \begin{cases} -\frac{1}{x} + C_1 & x > 0 \\ -\frac{1}{x} + C_2 & x < 0 \end{cases}$$

Example Find the indefinite integral:

$$\int x^3 + 2x + 5dx$$

Example Find the indefinite integral:

$$\int \frac{x^{5/2} + 5x^3 + 3}{x^2} dx$$

Example Find the indefinite integral

$$\int \frac{\sin x}{\cos^2 x} dx$$

Example Find the values of the definite integrals listed below:

$$\int_1^2 x^3 + 2x + 5dx, \qquad \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^2 x} dx, \qquad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x + 2\cos x dx$$

Interpretation and application of the Definite Integral

Recall that part 2 of the Fundamental Theorem of Calculus says that

$$\int_a^b f(x)dx = F(b) - F(a)$$

if F is an antiderivative for f and f is continuous on $[a, b]$. Since $F'(x) = f(x)$ for $a \leq x \leq b$, we can rewrite the theorem as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

when $F'(x)$ is continuous on $[a, b]$.

This sometimes make it easier to spot the definite integral in applications. $F(b) - F(a)$ is the **net change** in the function F on the interval $[a, b]$.
We can interpret the formula above as

$$\int_a^b \text{(Rate of change) } dx = \text{Net change between } a \text{ and } b.$$

Some common applications of the definite integral are as follows:

- If $V(t)$ is the volume of water that has passed through a pipe at time t, then $V'(t)$ is the rate of flow of water at time t and
$$\int_{t_1}^{t_2} V'(t)dt$$
is the volume of water that has passed through the pipe between time t_1 and time t_2.

- If the rate of growth of a population is given by $P'(t)$, then the net change in the population during the time period from t_1 to t_2 is given by
$$\int_{t_1}^{t_2} P'(t)dt = P(t_2) - P(t_1).$$

- If the cost of producing x units of a commodity is given by $C(x)$ and the marginal cost of producing x units of a commodity is $C'(x)$, then the increase in cost from raising production levels from $x = x_1$ to $x = x_2$ is
$$\int_{x_1}^{x_2} C'(x)dx = C(x_2) - C(x_1).$$

- If an object is moving along a straight line with position function $s(t)$ and velocity $s'(t) = v(t)$ then
$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$
is the net change in position or displacement of the object during the time period from t_1 to t_2.

- To calculate the distance that the object above travels during the time period from t_1 to t_2, we must integrate the speed function. The distance travelled by the object during the time period from t_1 to t_2 is
$$\int_{t_1}^{t_2} |v(t)|dt = \text{total distance travelled.}$$

Example A particle moves along a straight line. The velocity at time t is given by the $v(t) = t^2 - 4$ m/s.

(a) Find the displacement of the particle during the time period $0 < t < 3$.

(b) Find the distance travelled during this time period.

Example Water flows from a tank at the rate of $r(t) = 100 - 2t$ gallons per minute. How much water flows from the tank in the first 5 minutes?

Example The acceleration of a particle moving in a straight line is given by $a(t) = 2t + 1$ m/s^2. It is known that the initial velocity of the particle is $v(0) = 3$, find the velocity on the interval $0 \leq t \leq 10$ and find the distance travelled in the first 10 minutes.